

ON CLUSTER ALGEBRAS ARISING FROM UNPUNCTURED SURFACES

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ABSTRACT. We study cluster algebras that are associated to unpunctured surfaces, with coefficients arising from boundary arcs. We give a direct formula for the Laurent polynomial expansion of cluster variables in these cluster algebras in terms of certain paths on a triangulation of the surface. As an immediate consequence, we prove the positivity conjecture of Fomin and Zelevinsky for these cluster algebras. In the special case where the cluster algebra is acyclic, we also give a formula for the expansion of cluster variables as a polynomial whose indeterminates are the cluster variables contained in the union of an arbitrary acyclic cluster and all its neighbouring clusters in the mutation graph.

1. INTRODUCTION

Cluster algebras, introduced in [FZ1], are commutative algebras equipped with a distinguished set of generators, the *cluster variables*. The cluster variables are grouped into sets of constant cardinality n , the *clusters*, and the integer n is called the *rank* of the cluster algebra. Starting with an initial cluster \mathbf{x} (together with a skew symmetrizable integer $n \times n$ matrix $B = (b_{ij})$ and a coefficient vector $\mathbf{p} = (p_i^\pm)$ whose entries are elements of a torsion-free abelian group \mathbb{P}) the set of cluster variables is obtained by repeated application of so called *mutations*. To be more precise, let \mathcal{F} be the field of rational functions in the indeterminates x_1, x_2, \dots, x_n over the quotient field of the integer group ring $\mathbb{Z}\mathbb{P}$. Thus $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$ is a transcendence basis for \mathcal{F} . For every $k = 1, 2, \dots, n$, the mutation $\mu_k(\mathbf{x})$ of the cluster $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$ is a new cluster $\mu_k(\mathbf{x}) = \mathbf{x} \setminus \{x_k\} \cup \{x'_k\}$ obtained from \mathbf{x} by replacing the cluster variable x_k by the new cluster variable

$$(1) \quad x'_k = \frac{1}{x_k} \left(p_i^+ \prod_{b_{ki} > 0} x_i^{b_{ki}} + p_i^- \prod_{b_{ki} < 0} x_i^{-b_{ki}} \right)$$

in \mathcal{F} . Mutations also change the attached matrix B as well as the coefficient vector \mathbf{p} , see [FZ1].

The set of all cluster variables is the union of all clusters obtained from an initial cluster \mathbf{x} by repeated mutations. Note that this set may be infinite.

It is clear from the construction that every cluster variable is a rational function in the initial cluster variables x_1, x_2, \dots, x_n . In [FZ1] it is shown that every cluster variable u is actually a Laurent polynomial in the x_i , that is, u can be written as a reduced fraction

$$(2) \quad u = \frac{f(x_1, x_2, \dots, x_n)}{\prod_{i=1}^n x_i^{d_i}},$$

The first author is partly supported by the NSF grant DMS-0700358 and the University of Massachusetts at Amherst; the second author is partly supported by an NSERC Discovery Grant.

where $f \in \mathbb{Z}\mathbb{P}[x_1, x_2, \dots, x_n]$ and $d_i \geq 0$. The right hand side of equation (2) is called the *cluster expansion* of u in \mathbf{x} .

Inspired by the work of Fock and Goncharov [FG1, FG2, FG3] and Gekhtman, Shapiro and Vainshtein [GSV1, GSV2] which discovered cluster structures in the context of Teichmüller theory, Fomin, Shapiro and Thurston [FST] initiated a systematic study of the cluster algebras arising from triangulations of a surface with boundary and interior marked points. In this approach, clusters in the cluster algebra correspond to triangulations of the surface. Our first main result is a direct expansion formula for cluster variables in cluster algebras associated to unpunctured surfaces, with coefficients arising from boundary arcs, in terms of certain paths on the triangulation, see Theorem 3.2.

As an immediate consequence, we prove the positivity conjecture of Fomin and Zelevinsky [FZ1] for these cluster algebras, Corollary 3.6.

For *acyclic* cluster algebras, it has been shown in [BFZ1] that if the cluster $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$ occurs in an acyclic seed and if x'_1, x'_2, \dots, x'_n denote the n cluster variables obtained by mutating \mathbf{x} in each direction, then any cluster variable u can be written as a polynomial in $x_1, x_2, \dots, x_n, x'_1, x'_2, \dots, x'_n$, that is,

$$(3) \quad u = f(x_1, \dots, x_n, x'_1, \dots, x'_n) \quad \text{with } f \in \mathbb{Z}\mathbb{P}[x_1, \dots, x_n, x'_1, \dots, x'_n].$$

Our second main result, Theorem 5.3, is an explicit formula for this polynomial in the case of acyclic cluster algebras associated to unpunctured surfaces, with coefficients arising from boundary arcs.

Theorem 3.2 has interesting intersections with work of other people. In [CCS2], the authors obtained a formula for the denominators of the cluster expansion in types A , D and E , see also [BMR]. In [CC, CK, CK2] an expansion formula was given in the case where the cluster algebra is acyclic and the cluster lies in an acyclic seed. Palu recently generalized this formula to arbitrary clusters in an acyclic cluster algebra [Pa]. All these formulas use the cluster category introduced in [BMRRT], and in [CCS] for type A . The formula that we give in this paper not only uses a very different approach, it also covers a large variety of cluster algebras (parametrized by the genus of the surface, the number of boundary components and the number of marked points on the boundary) for which no formula has been known so far. The only surfaces that give rise to acyclic cluster algebras are the polygon and the annulus corresponding to the types A and \tilde{A} respectively. The proof of the positivity conjecture for arbitrary clusters is new even in the acyclic types; in [CK] and [CR] the conjecture is shown only in the case where the initial seed is acyclic.

In [SZ, CZ, Z, MP] cluster expansions for cluster algebras of rank 2 are given, in [Pr, CP, FZ3] the case A is considered and in [M] a cluster expansion for cluster algebras of finite type is given for clusters that lie in a bipartite seed.

The paper is organized as follows. In section 2, we recall the construction of [FST]. We state our expansion formula and give two examples in section 3. The proof of the formula given in section 4. Section 5 is devoted to our polynomial formula for acyclic clusters.

The work in this paper extends the work of the first author in [Sch], in which the expansion formula in the type A case was proved.

The authors would like to acknowledge helpful conversations with Gregg Musiker, Robert Marsh, and Sergey Fomin, and useful comments from a referee. They would also like to acknowledge the hospitality of the Centre de Recherches Mathématiques.

2. CLUSTER ALGEBRAS FROM SURFACES

In this subsection, we recall the construction of [FST] in the case of surfaces without punctures.

Let S be a connected oriented 2-dimensional Riemann surface with boundary and M a non-empty set of marked points in the closure of S with at least one marked point on each boundary component. The pair (S, M) is called *bordered surface with marked points*. Marked points in the interior of S are called *punctures*.

In this paper we will only consider surfaces (S, M) such that all marked points lie on the boundary of S , and we will refer to (S, M) simply by *unpunctured surface*.

We say that two curves in S *do not cross* if they do not intersect each other except that endpoints may coincide.

Definition 1. An arc γ in (S, M) is a curve in S such that

- (a) the endpoints are in M ,
- (b) γ does not cross itself,
- (c) the relative interior of γ is disjoint from M and from the boundary of S ,
- (d) γ does not cut out a monogon or a digon.

Curves that connect two marked points and lie entirely on the boundary of S without passing through a third marked point are called *boundary arcs*. Hence an arc is a curve between two marked points, which does not intersect itself nor the boundary except possibly at its endpoints and which is not homotopic to a point or a boundary arc.

Each arc is considered up to isotopy inside the class of such curves.

For any two arcs γ, γ' in S , let $e(\gamma, \gamma')$ be the minimal number of crossings of γ and γ' , that is, $e(\gamma, \gamma')$ is the minimum of the numbers of crossings of arcs α and α' , where α is isotopic to γ and α' is isotopic to γ' . Two arcs γ, γ' are called *compatible* if $e(\gamma, \gamma') = 0$. A *triangulation* is a maximal collection of compatible arcs together with all boundary arcs. The arcs of a triangulation cut the surface into *triangles*. Since (S, M) is an unpunctured surface, the three sides of each triangle are distinct (in contrast to the case of surfaces with punctures). Any triangulation has $n + m$ elements, n of which are arcs in S , and the remaining m elements are boundary arcs. Note that the number of boundary arcs is equal to the number of marked points.

Proposition 2.1. The number n of arcs in any triangulation is given by the formula $n = 6g + 3b + m - 6$, where g is the genus of S , b is the number of boundary components and $m = |M|$ is the number of marked points. The number n is called the *rank* of (S, M) .

Proof. [FST, 2.10] □

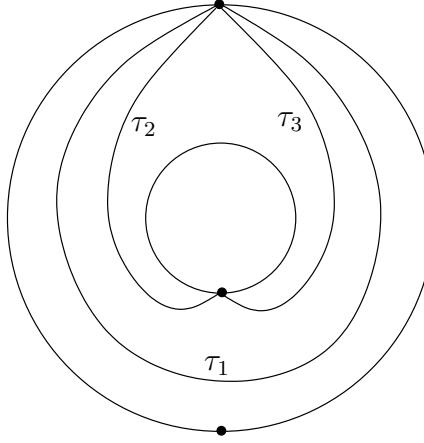
Note that $b > 0$ since the set M is not empty. Table 1 gives some examples of unpunctured surfaces.

Following [FST], we associate a cluster algebra $\mathcal{A}(S, M)$ to the unpunctured surface (S, M) as follows. The coefficient semifield is taken to be the tropical semifield $\text{Trop}(x_{n+1}, x_{n+2}, \dots, x_{n+m})$, which is a free abelian group, written multiplicatively, with m generators $x_{n+1}, x_{n+2}, \dots, x_{n+m}$, and with an auxiliary addition which we do not need to refer to here.

Choose any triangulation T , let $\tau_1, \tau_2, \dots, \tau_n$ be the n arcs of T and denote the m boundary arcs of the surface by $\tau_{n+1}, \tau_{n+2}, \dots, \tau_{n+m}$. Each of the boundary arcs

b	g	m	surface
1	0	$n+3$	polygon
1	1	$n-3$	torus with disk removed
1	2	$n-9$	genus 2 surface with disk removed
2	0	n	annulus
2	1	$n-6$	torus with 2 disks removed
2	2	$n-12$	genus 2 surface with 2 disks removed
3	0	$n-3$	pair of pants

TABLE 1. Examples of unpunctured surfaces

FIGURE 1. A triangulation with $b_{23} = 2$

is a side in precisely one triangle of the triangulation T . For any triangle Δ in T define a matrix $B^\Delta = (b_{ij}^\Delta)_{1 \leq i \leq m+n, 1 \leq j \leq n}$ by

$$b_{ij}^\Delta = \begin{cases} 1 & \text{if } \tau_i \text{ and } \tau_j \text{ are sides of } \Delta \\ & \text{with } \tau_j \text{ following } \tau_i \text{ in the clockwise order;} \\ -1 & \text{if } \tau_i \text{ and } \tau_j \text{ are sides of } \Delta, \\ & \text{with } \tau_j \text{ following } \tau_i \text{ in the counter-clockwise order;} \\ 0 & \text{otherwise.} \end{cases}$$

Then define the matrix $\tilde{B}(T) = (b_{ij})_{1 \leq i \leq n+m, 1 \leq j \leq n}$ by $b_{ij} = \sum_\Delta b_{ij}^\Delta$, where the sum is taken over all triangles in T , and let $B(T) = (b_{ij})_{1 \leq i, j \leq n}$ be the principal part of $\tilde{B}(T)$. The matrix $B(T)$ is skew-symmetric and each of its entries b_{ij} is either 0, 1, -1, 2, or -2. An example where $b_{ij} = 2$ is given in Figure 1.

Note that every arc τ can be in at most two triangles, since the surface S has no punctures.

Let $\mathcal{A}(S, M)$ be the cluster algebra given by the seed $(\mathbf{x}, \mathbf{p}, B(T))$ where $\mathbf{x} = \{x_{\tau_1}, x_{\tau_2}, \dots, x_{\tau_n}\}$ is the cluster associated to the triangulation T , and the initial coefficient vector $\mathbf{p} = (p_1^\pm, p_2^\pm, \dots, p_n^\pm) \in (\text{Trop}(x_{n+1}, x_{n+2}, \dots, x_{n+m}))^{2n}$ is given

by

$$p_j^+ = \prod_{i>n: b_{ij}=1} x_i \quad \text{and} \quad p_j^- = \prod_{i>n: b_{ij}=-1} x_i.$$

Remark 2.2. If one considers the cluster algebra $\mathcal{A}(S, M)$ with trivial coefficients then set $x_\tau = 1$ for each boundary arc τ .

3. CLUSTER EXPANSIONS

3.1. (T, γ) -paths. A path α in S is a continuous function $\alpha : [0, 1] \rightarrow S$. Let α and β be two paths in S , and let $x, y \in S$ be two points. Then we say that α and β are *homotopic between x and y* , if there exist $s_1, s_2, t_1, t_2 \in [0, 1]$ such that $s_1 < s_2$, $t_1 < t_2$, $\alpha(s_1) = \beta(t_1) = x$, $\alpha(s_2) = \beta(t_2) = y$ and the restrictions $\alpha|_{[s_1, s_2]} : [s_1, s_2] \rightarrow S$ and $\beta|_{[t_1, t_2]} : [t_1, t_2] \rightarrow S$ are homotopic as paths from x to y .

Let $T = \{\tau_1, \tau_2, \dots, \tau_n, \tau_{n+1}, \dots, \tau_N\}$, with $N = n + m$, be a triangulation of the unpunctured surface (S, M) , where τ_1, \dots, τ_n are arcs and $\tau_{n+1}, \dots, \tau_N$ are boundary arcs. Choose an orientation for each arc $\tau \in T$ and let $s(\tau)$ be the starting point of τ and $t(\tau)$ be its endpoint. Let τ^- be the arc τ with the opposite orientation. We will write τ^\pm if we want to consider both orientations at the same time. Let γ be an arc in (S, M) . Choose an orientation of γ and denote by a its starting point and by b its endpoint, thus $a, b \in M$. Let $k = \sum_{\tau \in T} e(\gamma, \tau)$ be the number of crossings between γ and T , and label the k crossing points of γ and T by $1, 2, \dots, k$ according to their order on γ such that 1 is the closest to a .

We will consider paths α in S that are concatenations of arcs in the triangulation T , more precisely, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)})$ with α_i or $\alpha_i^- \in T$, for $i = 1, 2, \dots, \ell(\alpha)$ and $s(\alpha_1) = s(\gamma)$, $t(\alpha_{\ell(\alpha)}) = t(\gamma)$, and $s(\alpha_i) = t(\alpha_{i-1})$, for $i = 2, \dots, \ell(\alpha)$. We call such a path a T -path. A T -path α is called *reduced* if $\alpha_i \neq \alpha_{i-1}^-$, for $i = 2, \dots, \ell(\alpha)$.

Definition 2. A (T, γ) -path α is a reduced T -path

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)})$$

such that

- (T1) $\ell(\alpha)$ is odd,
- (T2) if i is even, then α_i crosses γ ,
- (T3) for any $\tau \in T$, the number of even integers i such that $\alpha_i = \tau^\pm$ is at most $e(\tau, \gamma)$,
- (T4) there exists a sequence $\mathbf{i}_\alpha = (i_0, i_2, i_4, \dots, i_{\ell(\alpha)-1}, i_{\ell(\alpha)+1})$, $i_j \in \{1, 2, \dots, k\}$, such that $i_0 = s(\gamma)$, $i_{\ell(\alpha)+1} = t(\gamma)$ and $(i_2, i_4, \dots, i_{\ell(\alpha)-1})$ is a sequence of labeled crossing points such that $i_j < i_\ell$ if $j < \ell$ and the crossing point i_j lies on α_j , for $j = 2, 4, 6, \dots, \ell(\alpha) - 1$,
- (T5) for any two points i_j, i_ℓ in the sequence \mathbf{i}_α in (T4), with $j < \ell$, the paths α and γ are homotopic between the points i_j and i_ℓ .

To any (T, γ) -path $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)})$, we associate an element $x(\alpha)$ in the cluster algebra $\mathcal{A}(S, M)$ by

$$(4) \quad x(\alpha) = \prod_{i \text{ odd}} x_{\alpha_i} \prod_{i \text{ even}} x_{\alpha_i}^{-1}.$$

Note that $x_\tau = x_{\tau^-}$.

Definition 3. Let $\mathcal{P}_T(\gamma)$ denote the set of (T, γ) -paths.

Proposition 3.1. *If S is simply connected, then*

- (a) *for any two arcs α and β in S , we have $e(\alpha, \beta) \leq 1$,*
- (b) *any arc in S can occur at most once in any (T, γ) -path,*
- (c) *the map $\mathcal{P}_T(\gamma) \rightarrow \mathcal{A}(S, M)$, $\alpha \mapsto x(\alpha)$ is injective.*

Proof. Let T be a triangulation, γ an arc and $\alpha \in \mathcal{P}_T(\gamma)$.

(a) This follows directly from the definition of $e(\alpha, \beta)$ and the fact that S is simply connected.

(b) Suppose that $\alpha_j = \alpha_\ell^\pm$ with $j \neq \ell$. Then α contains a loop α° which runs through at least one crossing point i_t of γ and T , and i_t lies on an arc $\tau \in T$. Statement (a) implies that i_t is the only crossing point on τ . The loop α° is homotopically trivial, since S is simply connected. By condition (T5), α and γ are homotopic between $a = s(\gamma)$ and $b = t(\gamma)$, and then the isotopy class of γ contains an arc which does not cross τ , hence i_t is not a crossing point, a contradiction. This shows (b).

(c) Suppose that S is simply connected and $x(\alpha) = x(\alpha')$. Then (b) implies that the set of even arcs and the set of odd arcs are the same up to orientation in α and α' . From condition (T4) it follows that the order of the even arcs is the same. The even arcs divide S into regions. There is a unique odd arc in the region between successive even arcs. Therefore the order of the odd arcs must be the same, and thus the order of the marked points along the paths are the same. By simply-connectedness, knowing the sequence of vertices determines the paths, and thus the two paths are the same. \square

3.2. Expansion formula. The following theorem is the main result of this section.

Theorem 3.2. *Let T be any triangulation of an unpunctured surface (S, M) . Let γ be any arc in (S, M) and let x_γ denote the corresponding cluster variable in $\mathcal{A}(S, M)$. Then*

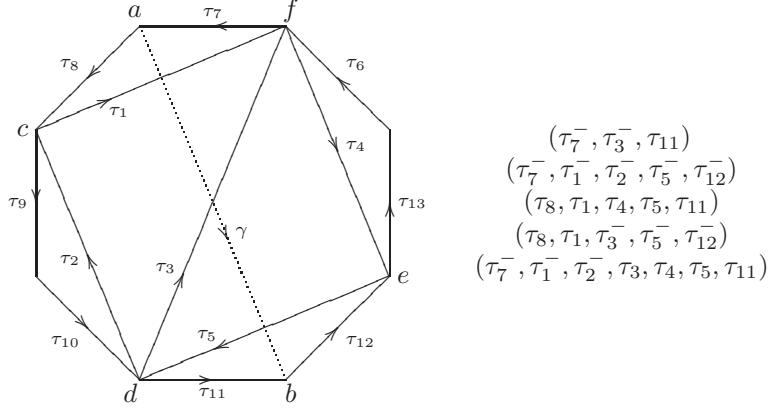
$$(5) \quad x_\gamma = \sum_{\alpha \in \mathcal{P}_T(\gamma)} x(\alpha).$$

Remark 3.3. If S is simply connected, then Proposition 3.1 implies that each $x(\alpha)$ is a reduced fraction whose denominator is a product of cluster variables, and that each term in the sum of equation (5) appears with multiplicity one.

The proof of Theorem 3.2 will be given in section 4. To illustrate the statement, we give two examples here.

Example 3.4. The case A_n : *The cluster algebra $\mathcal{A}(S, M)$ is of type A_n if (S, M) is an $(n+3)$ -gon. Our example illustrates the case $n = 5$. The following figure shows a triangulation $T = \{\tau_1, \dots, \tau_{13}\}$ and a (dotted) arc γ . Next to it is a complete list*

of elements of $\mathcal{P}_T(\gamma)$.



Theorem 3.2 thus implies that

$$x_\gamma = \frac{x_7x_{11}}{x_3} + \frac{x_7x_2x_{12}}{x_1x_5} + \frac{x_8x_4x_{11}}{x_1x_5} + \frac{x_8x_3x_{12}}{x_1x_5} + \frac{x_7x_2x_4x_{11}}{x_1x_3x_5}.$$

Example 3.5. The case \tilde{A}_{n-1} : The cluster algebra $\mathcal{A}(S, M)$ is of type \tilde{A}_{n-1} if (S, M) is an annulus. Our example illustrates the case $n = 4$. Figure 2 shows a triangulation $T = \{\tau_1, \tau_2, \dots, \tau_8\}$ and a (dotted) arc γ . The complete list of elements of $\mathcal{P}_T(\gamma)$ is as follows:

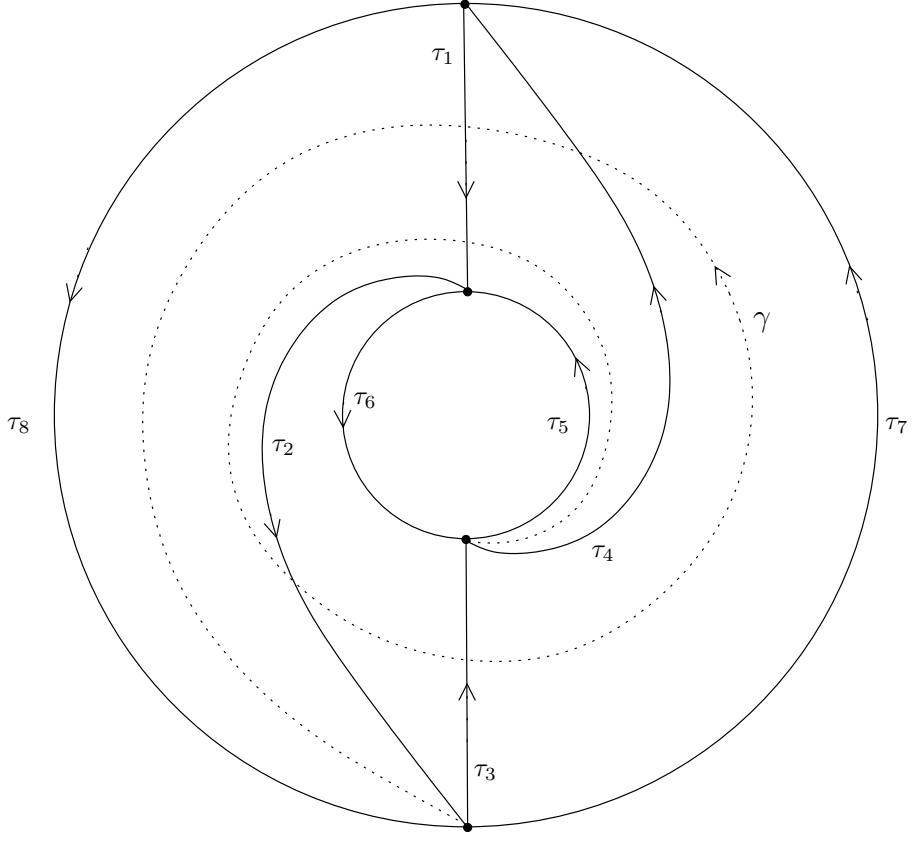
$$\begin{array}{ll}
 (\tau_5, \tau_1^-, \tau_8, \tau_3, \tau_4, \tau_1, \tau_2) & (\tau_5, \tau_1^-, \tau_8, \tau_3, \tau_5, \tau_1^-, \tau_8) \\
 (\tau_5, \tau_1^-, \tau_8, \tau_2^-, \tau_6, \tau_4, \tau_8) & (\tau_5, \tau_1^-, \tau_8, \tau_2^-, \tau_6, \tau_3^-, \tau_7, \tau_1, \tau_2) \\
 (\tau_5, \tau_1^-, \tau_8, \tau_2^-, \tau_6, \tau_3^-, \tau_7, \tau_4^-, \tau_5, \tau_1^-, \tau_8) & (\tau_5, \tau_2, \tau_3, \tau_4, \tau_8) \\
 (\tau_5, \tau_2, \tau_7, \tau_1, \tau_2) & (\tau_5, \tau_2, \tau_7, \tau_4^-, \tau_5, \tau_1^-, \tau_8) \\
 (\tau_4, \tau_1, \tau_2, \tau_3, \tau_4, \tau_1, \tau_2) & (\tau_4, \tau_1, \tau_2, \tau_3, \tau_5, \tau_1^-, \tau_8) \\
 (\tau_4, \tau_1, \tau_6, \tau_3^-, \tau_7, \tau_1, \tau_2) & (\tau_4, \tau_1, \tau_6, \tau_3^-, \tau_7, \tau_4^-, \tau_5, \tau_1^-, \tau_8) \\
 (\tau_4, \tau_1, \tau_6, \tau_4, \tau_8) &
 \end{array}$$

Hence Theorem 3.2 implies

$$\begin{aligned}
 x_\gamma = & \frac{x_5x_8x_4x_2}{x_1x_3x_1} + \frac{x_5x_8x_5x_8}{x_1x_3x_1} + \frac{x_5x_8x_6x_8}{x_1x_2x_4} + \frac{x_5x_8x_6x_7x_2}{x_1x_2x_3x_1} + \frac{x_5x_8x_6x_7x_5x_8}{x_1x_2x_3x_4x_1} \\
 & + \frac{x_5x_3x_8}{x_2x_4} + \frac{x_5x_7x_2}{x_2x_1} + \frac{x_5x_7x_5x_8}{x_2x_4x_1} + \frac{x_4x_2x_4x_2}{x_1x_3x_1} + \frac{x_4x_2x_5x_8}{x_1x_3x_1} + \frac{x_4x_6x_7x_2}{x_1x_3x_1} \\
 & + \frac{x_4x_6x_7x_5x_8}{x_1x_3x_4x_1} + \frac{x_4x_6x_8}{x_1x_4},
 \end{aligned}$$

which can be simplified to

$$\begin{aligned}
 x_\gamma = & 2\frac{x_2x_4x_5x_8}{x_1^2x_3} + \frac{x_5^2x_8^2}{x_1^2x_3} + \frac{x_5x_6x_8^2}{x_1x_2x_4} + 2\frac{x_5x_6x_7x_8}{x_1^2x_3} + \frac{x_5^2x_6x_7x_8^2}{x_1^2x_2x_3x_4} \\
 & + \frac{x_3x_5x_8}{x_2x_4} + \frac{x_5x_7}{x_1} + \frac{x_5^2x_7x_8}{x_1x_2x_4} + \frac{x_2^2x_4^2}{x_1^2x_3} + \frac{x_2x_4x_6x_7}{x_1^2x_3} + \frac{x_6x_8}{x_1}.
 \end{aligned}$$

FIGURE 2. The case \tilde{A}_{n-1}

3.3. Positivity. The following positivity conjecture of [FZ1] is a direct consequence of Theorem 3.2.

Corollary 3.6. *Let (S, M) be an unpunctured surface. Let x be any cluster variable in the cluster algebra $\mathcal{A}(S, M)$, and let $\{x_1, \dots, x_n\}$ be any cluster. Let*

$$x = \frac{f(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m})}{x_1^{d_1} \dots x_n^{d_n}}$$

be the expansion of x in the cluster $\{x_1, \dots, x_n\}$, where f is a polynomial which is not divisible by any of the x_1, \dots, x_n . Then

- (a) *the coefficients of f are non-negative integers,*
- (b) *if S is simply connected, the coefficients of f are either 0 or 1.*

Proof. (a) is a direct consequence of Theorem 3.2, and (b) follows from Remark 3.3. \square

4. PROOF OF THEOREM 3.2

4.1. The simply connected case. Assume that S is simply connected. Let $T = \{\tau_1, \dots, \tau_N\}$, with $N = n + m$, be a triangulation of S and let γ be an arc in S .

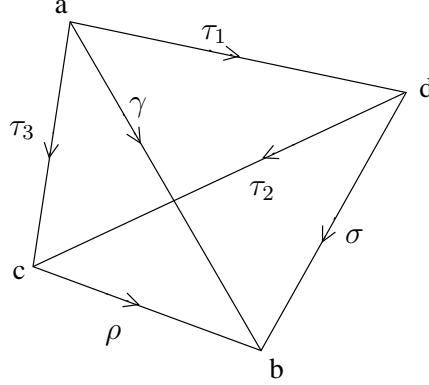


FIGURE 3. Proof of Theorem 3.2

Choose an orientation of γ and let a be its starting point and b be its endpoint. Suppose that $\gamma \notin T$. Proposition 3.1 implies that every arc in T crosses γ at most once. Among all arcs of T that cross γ , there is a unique one, say τ_2 , such that its crossing point with γ is the closest possible to the vertex a . Then there is a unique triangle in T having τ_2 as one side and the vertex a as third point. Denote the other two sides of this triangle by τ_1 and τ_3 and let c be the common endpoint of τ_3 and τ_2 , and d the common endpoint of τ_1 and τ_2 (see Figure 3). Note that τ_1, τ_3 may be boundary arcs. Now consider the unique quadrilateral in which γ and τ_2 are the diagonals. Two of its sides are τ_1 and τ_3 . Denote the other two sides by ρ and σ in such a way that ρ is the side opposite to τ_1 (see Figure 3). We assume without loss of generality that the orientations of the arcs $\tau_1, \tau_2, \tau_3, \rho$ and σ are as in Figure 3. In particular $s(\tau_1) = s(\tau_3) = a$ and $s(\tau_2) = t(\tau_1)$. We will keep this setup for the rest of this subsection.

Lemma 4.1. (a) *If $\tau_i \in T$ crosses ρ (respectively σ), then τ_i crosses γ .*
 (b) *If $\tau_i \in T$ is incident to a , then τ_i crosses neither ρ nor σ .*
 (c) *If $\tau_i \in T$ crosses γ and does not cross ρ (respectively σ), then τ_i is incident to c (respectively d).*

Proof. Since S is simply connected, this follows directly from the construction and the fact that τ_i does not cross τ_2 . \square

Let $\mathcal{P}_T(\gamma)_{\tau_j}$ denote the subset of $\mathcal{P}_T(\gamma)$ of all (T, γ) -paths α that start with the arc τ_j^\pm and let $\mathcal{P}_T(\gamma)_{-\tau_j}$ be the subset of $\mathcal{P}_T(\gamma)$ of all (T, γ) -paths α that do not contain the arc τ_j^\pm . Similarly, let $\mathcal{P}_T(\gamma)_{\tau_j \tau_\ell}$ denote the subset of $\mathcal{P}_T(\gamma)_{\tau_j}$ of all (T, γ) -paths α that start with the arcs $\tau_j^\pm \tau_\ell^\pm$ and let $\mathcal{P}_T(\gamma)_{\tau_j, -\tau_\ell}$ be the subset of $\mathcal{P}_T(\gamma)_{\tau_j}$ of all (T, γ) -paths α that start with the arc τ_j^\pm and do not contain the arc τ_ℓ^\pm .

Lemma 4.2. *We have $\mathcal{P}_T(\gamma) = \mathcal{P}_T(\gamma)_{\tau_1} \sqcup \mathcal{P}_T(\gamma)_{\tau_3}$.*

Proof. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)})$ be an arbitrary element of $\mathcal{P}_T(\gamma)$. Then $s(\alpha_1) = a$, and α_2 crosses γ at the crossing point i_2 of \mathbf{i}_α . Condition (T5) implies that α and γ are homotopic between a and i_2 . On the other hand, γ crosses τ_2 at the crossing point $1 \leq i_2$, and thus, α_1 must intersect τ_2 either in the interior of S

or on the boundary. Therefore, either $\alpha_2 = \tau_2$ or $t(\alpha_1) \in \{c, d\}$, and in both cases we must have $\alpha_1 \in \{\tau_1^\pm, \tau_3^\pm\}$. \square

Lemma 4.3. *We have*

- (a) $\mathcal{P}_T(\gamma)_{\tau_1} = \mathcal{P}_T(\gamma)_{\tau_1 \tau_2} \sqcup \mathcal{P}_T(\gamma)_{\tau_1, -\tau_2}$,
- (b) $\mathcal{P}_T(\gamma)_{\tau_3} = \mathcal{P}_T(\gamma)_{\tau_3 \tau_2} \sqcup \mathcal{P}_T(\gamma)_{\tau_3, -\tau_2}$.

Proof. By construction, τ_2 is the arc of the triangulation T such that its crossing point with γ is the closest possible to the vertex a . Hence, the result follows from condition (T4). \square

Let ρ be as above (see Figure 3). We will construct two maps $f : \mathcal{P}_T(\rho)_{\tau_2} \rightarrow \mathcal{P}_T(\gamma)$ and $g : \mathcal{P}_T(\rho)_{-\tau_2} \rightarrow \mathcal{P}_T(\gamma)$.

Let $\beta = (\beta_1, \dots, \beta_{\ell(\beta)})$ be any path in $\mathcal{P}_T(\rho)$. Suppose first that $\beta_1 = \tau_2^-$, thus $t(\beta_1) = d$. In this case, let $f(\beta)$ be the path in $\mathcal{P}_T(\gamma)$ obtained from β by replacing the first arc τ_2^- by τ_1 , that is

$$f(\beta) = (\tau_1, \beta_2, \dots, \beta_{\ell(\beta)}).$$

Suppose now that $\beta_j \neq \tau_2^\pm$ for all j . In this case, let $g(\beta)$ be the composition of the paths (τ_1, τ_2) and β , that is

$$g(\beta) = (\tau_1, \tau_2, \beta_1, \beta_2, \dots, \beta_{\ell(\beta)}).$$

Let us check that $f(\beta)$ and $g(\beta)$ are elements of $\mathcal{P}_T(\gamma)$. Indeed, the fact that $f(\beta)$ and $g(\beta)$ are T -paths is immediate; they are reduced by construction, property (T1) is clear and (T2) follows from Lemma 4.1(a). In order to show (T3), we need to prove that $\beta_j \neq \tau_2^\pm$, for all even j ; but this follows from the fact that for even j , β_j crosses ρ . The condition (T4) holds since the crossing point of τ_2 and γ is the closest possible to a , and (T5) holds, since S is simply connected. We have the following lemma.

Lemma 4.4. *The maps f and g induce bijections*

$$f : \mathcal{P}_T(\rho)_{\tau_2} \rightarrow \mathcal{P}_T(\gamma)_{\tau_1, -\tau_2} \quad \text{and} \quad g : \mathcal{P}_T(\rho)_{-\tau_2} \rightarrow \mathcal{P}_T(\gamma)_{\tau_1 \tau_2},$$

and

$$(6) \quad x(f(\gamma)) = \frac{x_{\tau_1}}{x_{\tau_2}} x(\gamma) \quad \text{and} \quad x(g(\gamma)) = \frac{x_{\tau_1}}{x_{\tau_2}} x(\gamma)$$

Proof. The formulas (6) follow directly from the definitions of f and g . These formulas together with Proposition 3.1 imply the injectivity of f and g . To show the surjectivity of f , suppose that $\mathcal{P}_T(\gamma)_{\tau_1, -\tau_2}$ is not empty and let $\alpha \in \mathcal{P}_T(\gamma)_{\tau_1, -\tau_2}$ be an arbitrary element. Say

$$\alpha = (\tau_1, \alpha_2, \alpha_3, \alpha_4, \dots, \alpha_{\ell(\alpha)}).$$

We need to show that the path

$$\beta = (\tau_2^-, \alpha_2, \alpha_3, \alpha_4, \dots, \alpha_{\ell(\alpha)})$$

is an element of $\mathcal{P}_T(\rho)_{\tau_2}$. It is a reduced T -path because the path α does not contain the arc τ_2^\pm . Condition (T1) holds since $\alpha \in \mathcal{P}_T(\gamma)$ and condition (T5) holds since S is simply connected. Moreover, since $\mathcal{P}_T(\gamma)_{\tau_1, -\tau_2}$ is not empty, there exists an arc in $T \setminus \{\tau_2\}$ which is incident to d and crosses γ . Since T is a triangulation, it follows that any diagonal in $T \setminus \{\tau_2\}$ that crosses γ also crosses ρ . Thus β satisfies conditions (T2) and (T4), because $\alpha \in \mathcal{P}_T(\gamma)$. Consequently, β also satisfies condition (T3),

by Proposition 3.1 (b). This shows that $\beta \in \mathcal{P}_T(\rho)$, and since β starts with the arc τ_2^- , we have $\beta \in \mathcal{P}_T(\rho)_{\tau_2^-}$. Hence f is surjective.

It remains to show that g is surjective. Let $\alpha \in \mathcal{P}_T(\gamma)_{\tau_1 \tau_2}$ be arbitrary. Say

$$\alpha = (\tau_1, \tau_2, \alpha_3, \alpha_4, \dots, \alpha_{\ell(\alpha)}).$$

We have to show that

$$\beta = (\alpha_3, \alpha_4, \dots, \alpha_{\ell(\alpha)}) \in \mathcal{P}_T(\rho).$$

It is a reduced T -path, since $\alpha \in \mathcal{P}_T(\gamma)$, condition (T1) follows since $s(\alpha_3) = t(\tau_2) = c$, and condition (T5) holds because S is simply connected. Let us show (T2). We need to show that any even arc of β crosses ρ . Since $\alpha \in \mathcal{P}_T(\gamma)$, we know that every even arc of β crosses γ . Thus by Lemma 4.1(c), if there is an even arc of β that does not cross ρ , then this arc has to be incident to c . Since β starts at c , its first even arc α_4 cannot be incident to c , because α is reduced, and thus α_4 crosses both γ and ρ . Then, since α satisfies (T4), every even arc of β crosses γ and ρ . This shows (T2), and (T4) follows from the fact that $\alpha \in \mathcal{P}_T(\gamma)$. Condition (T3) holds by Proposition 3.1. Hence $\beta \in \mathcal{P}_T(\rho)$ and g is surjective. \square

Lemma 4.5. *We have*

$$(a) \quad \sum_{\beta \in \mathcal{P}_T(\rho)} x(\gamma) \frac{x_{\tau_1}}{x_{\tau_2}} = \sum_{\alpha \in \mathcal{P}_T(\gamma)_{\tau_1}} x(\alpha)$$

$$(b) \quad \sum_{\beta \in \mathcal{P}_T(\sigma)} x(\gamma) \frac{x_{\tau_3}}{x_{\tau_2}} = \sum_{\alpha \in \mathcal{P}_T(\gamma)_{\tau_3}} x(\alpha).$$

Proof. The first statement follows from Lemma 4.3(a), Lemma 4.4 and the fact that $\mathcal{P}_T(\rho) = \mathcal{P}_T(\rho)_{\tau_2} \sqcup \mathcal{P}_T(\rho)_{-\tau_2}$. The second statement follows by symmetry. \square

Proof of Theorem 3.2 for simply connected surfaces S . The total number of crossings between γ and T is $e(T, \gamma) = \sum_{\tau_i \in T} e(\tau_i, \gamma)$.

We prove the theorem by induction on $e(T, \gamma)$. If $e(T, \gamma) = 0$, then $\gamma \in T$. In this case, no element of T crosses γ and, by condition (T2), the set $\mathcal{P}_T(\gamma)$ contains exactly one element: $\mathcal{P}_T(\gamma) = \{\gamma\}$. Thus

$$\sum_{\alpha \in \mathcal{P}_T(\gamma)} x(\alpha) = x(\gamma) = x_\gamma.$$

Suppose now that $e(T, \gamma) \geq 1$. As before, consider the unique quadrilateral in which γ and τ_2 are the diagonals (see Figure 3). Thus, in the cluster algebra $\mathcal{A}(S, M)$, we have the exchange relation

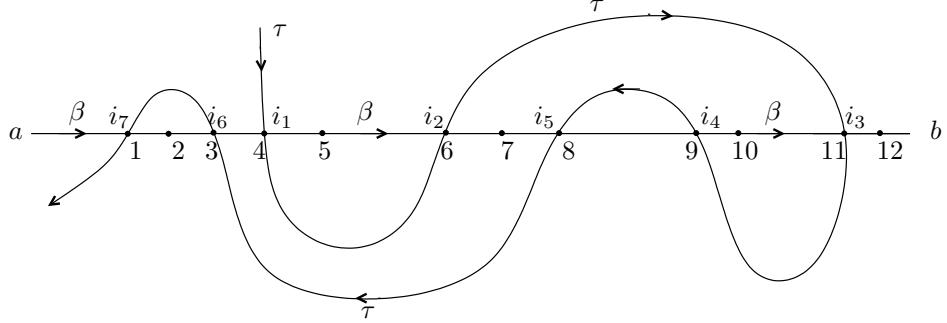
$$(7) \quad x_\gamma x_{\tau_2} = x_{\tau_1} x_\rho + x_{\tau_3} x_\sigma.$$

Moreover, any arc in T that crosses ρ (respectively σ) also crosses γ , by Lemma 4.1(a), and, moreover, τ_2 crosses γ but crosses neither ρ nor σ . Thus $e(T, \rho) < e(T, \gamma)$ and $e(T, \sigma) < e(T, \gamma)$, and by induction hypothesis

$$x_\rho = \sum_{\beta \in \mathcal{P}_T(\rho)} x(\beta) \quad \text{and} \quad x_\sigma = \sum_{\beta \in \mathcal{P}_T(\sigma)} x(\beta).$$

Therefore, we can write the exchange relation (7) as

$$x_\gamma = \sum_{\beta \in \mathcal{P}_T(\rho)} x(\beta) \frac{x_{\tau_1}}{x_{\tau_2}} + \sum_{\beta \in \mathcal{P}_T(\sigma)} x(\beta) \frac{x_{\tau_3}}{x_{\tau_2}}.$$

FIGURE 4. Proof of Lemma 4.6, $k = 12$ and $i_\ell = i_2$.

The theorem now follows from Lemma 4.5 and Lemma 4.2. \square

4.2. The non-simply connected case. In this subsection, we prove Theorem 3.2 for any unpunctured surface (S, M) . The key idea of the proof is to work in a universal cover of S , so that we can use Theorem 3.2 for simply connected surfaces. First, we prove a Lemma which will allow us to use induction later.

Lemma 4.6. *Let $T = \{\tau_1, \dots, \tau_N\}$ be a triangulation of S , and let β be an arc in S which is not in T . Let $e(\beta, T)$ be the number of crossings between β and T . Then there exist five arcs $\rho_1, \rho_2, \sigma_1, \sigma_2$ and β' in S such that*

- (a) *each of $\rho_1, \rho_2, \sigma_1, \sigma_2$ and β' crosses T less than $e(\beta, T)$ times,*
- (b) *$\rho_1, \rho_2, \sigma_1, \sigma_2$ are the sides of a simply connected quadrilateral in which β and β' are the diagonals,*
- (c) *in the cluster algebra $\mathcal{A}(S, M)$, we have the exchange relation*

$$(8) \quad x_\beta x_{\beta'} = x_{\rho_1} x_{\rho_2} + x_{\sigma_1} x_{\sigma_2}.$$

Proof. We prove the Lemma by induction on $k = e(\beta, T)$. If $k = 1$, then let $\beta' \in T$ be the unique arc that crosses β . Then there exists a unique quadrilateral in T in which β' and β are the diagonals. Let ρ_1 and ρ_2 be two opposite sides of this quadrilateral and let σ_1 and σ_2 be the other two opposite sides. These arcs satisfy (a), (b) and (c).

Suppose that $k \geq 2$. Choose an orientation of β and denote its starting point by a and its endpoint by b (note that a and b may be the same point). Label the k crossing points of β and T by $1, 2, \dots, k$ according to their order on β , such that the crossing point 1 is the closest to a . Let τ be an arc of the triangulation T such that $e(\tau, \beta) = r \geq 1$. As before with β , label the r crossing points of τ and β by i_1, i_2, \dots, i_r according to their order on τ ! (see Figure 4). Thus $r \leq k$, $\{i_1, i_2, \dots, i_r\} \subset \{1, 2, \dots, k\}$. Note that $j < \ell$ does not imply $i_j < i_\ell$.

Using τ and β , we will now construct the five arcs of the Lemma. Recall that β^- (respectively τ^-) denotes the path β (respectively τ) with the opposite orientation.

Let ℓ be such that

$$(9) \quad |(k+1)/2 - i_\ell| \leq |(k+1)/2 - i_j|, \quad \text{for all } j = 1, 2, \dots, r.$$

In other words, among all r crossing points of β and τ , the point i_ℓ is midmost on β with respect to the k crossing points of β and T (see Figure 4 for an example). We will distinguish four cases:

(1) $i_{\ell-1} < i_\ell$ and $i_{\ell+1} < i_\ell$. Consider the following arcs (see Figure 5):

Let $\beta' = (a, i_{\ell-1}, i_{\ell+1}, a \mid \beta, \tau, \beta^-)$ be the arc that starts at a and is homotopic to β up to the crossing point $i_{\ell-1}$, then, from $i_{\ell-1}$ to $i_{\ell+1}$, β' is homotopic to τ , and from $i_{\ell+1}$ to a , β' is homotopic to β^- . Note that β' and β cross exactly once, namely at the point i_ℓ .

In a similar way, let

$$\begin{aligned} \rho_1 &= (a, i_\ell, i_{\ell-1}, a \mid \beta, \tau^-, \beta^-) & \rho_2 &= (b, i_\ell, i_{\ell+1}, a \mid \beta^-, \tau, \beta^-) \\ \sigma_1 &= (a, i_{\ell-1}, i_\ell, b \mid \beta, \tau, \beta) & \sigma_2 &= (a, i_{\ell+1}, i_\ell, a \mid \beta, \tau^-, \beta^-) \end{aligned}$$

In the special case where $\ell = 1$, (respectively $l = r$), we define

$$\begin{aligned} \beta' &= (c, i_{\ell+1}, a \mid \tau, \beta^-) & \text{(respectively } \beta' = (a, i_{\ell-1}, d \mid \beta, \tau) \text{)} \\ \rho_1 &= (a, i_\ell, c \mid \beta, \tau^-) & \text{(respectively } \rho_2 = (b, i_\ell, d \mid \beta^-, \tau) \text{)} \\ \sigma_1 &= (c, i_\ell, b \mid \tau, \beta) & \text{(respectively } \sigma_2 = (d, i_\ell, a \mid \tau^-, \beta^-) \text{),} \end{aligned}$$

where c is the starting point of τ and d is its endpoint.

Note that, since $i_{\ell-1}, i_\ell$ and $i_{\ell+1}$ are distinct crossing points of β and T , the paths $(i_{\ell-1}, i_\ell, i_{\ell-1} \mid \tau, \beta^-)$ and $(i_{\ell+1}, i_\ell, i_{\ell+1} \mid \tau^-, \beta)$ are not homotopically trivial.

Then $\rho_1, \sigma_1, \rho_2, \sigma_2$ form a simply connected quadrilateral such that ρ_1 and ρ_2 are opposite sides, σ_1 and σ_2 are opposite sides, and β and β' are the diagonals. This shows (b). Moreover, since β and β' cross exactly once, we have the exchange relation in (c). It remains to show (a). Using the hypothesis $i_{\ell-1} < i_\ell$ and $i_{\ell+1} < i_\ell$, and the inequality (9), we get $i_{\ell-1} \leq (k+1)/2, i_{\ell+1} \leq (k+1)/2, i_\ell + i_{\ell-1} \leq k+1$ and $i_\ell + i_{\ell+1} \leq k+1$. Thus

$$\begin{aligned} e(\beta', T) &= (i_{\ell-1} - 1) + (i_{\ell+1} - 1) < k \\ e(\rho_1, T) &= (i_{\ell-1} - 1) + (i_\ell - 1) < k \\ e(\rho_2, T) &= (i_{\ell+1} - 1) + (k - i_\ell) < k + (i_{\ell+1} - i_\ell - 1) < k \\ e(\sigma_1, T) &= (i_{\ell-1} - 1) + (k - i_\ell) < k + (i_{\ell-1} - i_\ell - 1) < k \\ e(\sigma_2, T) &= (i_{\ell+1} - 1) + (i_\ell - 1) < k. \end{aligned}$$

In the case where $\ell = 1$, we have

$$\begin{aligned} e(\beta', T) &= i_{\ell+1} - 1 < k \\ e(\rho_1, T) &= i_\ell - 1 < k \\ e(\sigma_1, T) &= k - i_\ell < k, \end{aligned}$$

and in the case where $\ell = r$, we have

$$\begin{aligned} e(\beta', T) &= i_{\ell-1} - 1 < k \\ e(\rho_2, T) &= k - i_\ell < k \\ e(\sigma_2, T) &= i_\ell - 1 < k. \end{aligned}$$

This shows (a).

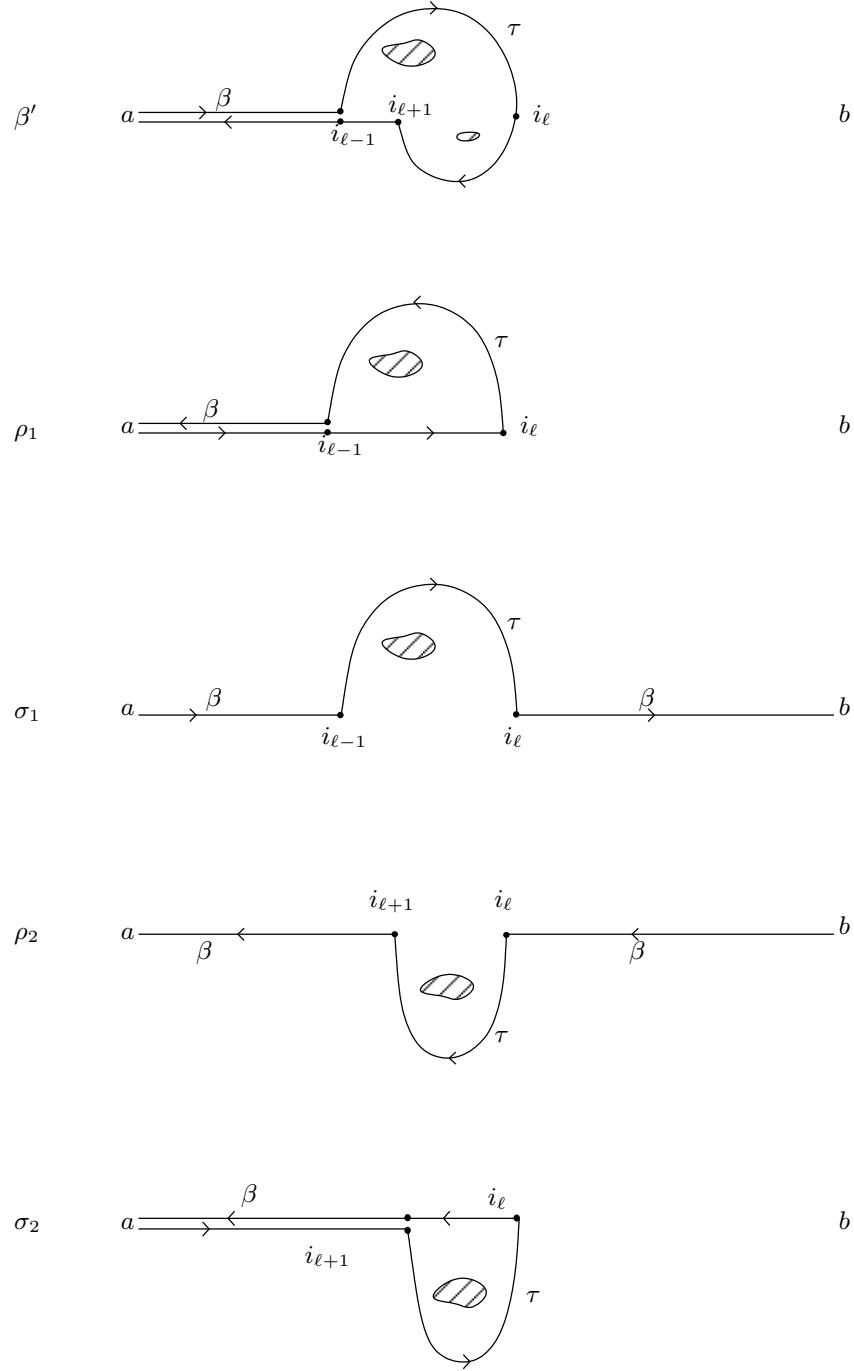


FIGURE 5. The arcs of Lemma 4.6 in case where $i_{\ell-1} < i_{\ell}$ and $i_{\ell+1} < i_{\ell}$.

(2) $i_{\ell-1} < i_\ell$ and $i_{\ell+1} > i_\ell$. Consider the following arcs, see Figure 6:

$$\begin{aligned}\beta' &= (a, i_{\ell-1}, i_{\ell+1}, b \mid \beta, \tau, \beta) \\ \rho_1 &= (a, i_{\ell-1}, i_\ell, a \mid \beta, \tau, \beta^-) \quad \rho_2 = (b, i_{\ell+1}, i_\ell, b \mid \beta^-, \tau^-, \beta) \\ \sigma_1 &= (a, i_\ell, i_{\ell+1}, b \mid \beta, \tau, \beta) \quad \sigma_2 = (b, i_\ell, i_{\ell-1}, a \mid \beta^-, \tau^-, \beta^-).\end{aligned}$$

In the special case where $\ell = 1$, (respectively $l = r$), we define

$$\begin{aligned}\beta' &= (c, i_{\ell+1}, b \mid \tau, \beta) \quad (\text{respectively } \beta' = (a, i_{\ell-1}, d \mid \beta, \tau)) \\ \rho_1 &= (c, i_\ell, a \mid \tau, \beta) \quad (\text{respectively } \rho_2 = (d, i_\ell, b \mid \tau^-, \beta)) \\ \sigma_2 &= (b, i_\ell, c \mid \beta, \tau) \quad (\text{respectively } \sigma_1 = (a, i_\ell, d \mid \beta^-, \tau^-)),\end{aligned}$$

where c is the starting point of τ and d is its endpoint.

Again $\rho_1, \sigma_1, \rho_2, \sigma_2$ form a simply connected quadrilateral such that ρ_1 and ρ_2 are opposite sides, σ_1 and σ_2 are opposite sides, and β and β' are the diagonals. This shows (b). Moreover, since β and β' cross exactly once, we have the exchange relation in (c). It remains to show (a). Using the hypothesis $i_{\ell-1} < i_\ell$ and $i_{\ell+1} > i_\ell$, and the inequality (9), we get $i_{\ell-1} \leq (k+1)/2, k - i_{\ell+1} \leq (k+1)/2, i_\ell + i_{\ell-1} \leq k+1$ and $i_\ell + i_{\ell+1} \geq k+1$. Thus

$$\begin{aligned}e(\beta', T) &= (i_{\ell-1} - 1) + (k - i_{\ell+1}) < k \\ e(\rho_1, T) &= (i_{\ell-1} - 1) + (i_\ell - 1) < k \\ e(\rho_2, T) &= (k - i_{\ell+1}) + (k - i_\ell) \leq 2k - (i_\ell + i_{\ell+1}) \leq 2k - (k+1) < k \\ e(\sigma_1, T) &= (i_\ell - 1) + (k - i_{\ell+1}) < k \\ e(\sigma_2, T) &= (k - i_\ell) + (i_{\ell-1} - 1) < k.\end{aligned}$$

In the case where $\ell = 1$, we have

$$\begin{aligned}e(\beta', T) &= k - i_{\ell+1} < k \\ e(\rho_1, T) &= i_\ell - 1 < k \\ e(\sigma_2, T) &= k - i_\ell < k,\end{aligned}$$

and in the case where $\ell = r$, we have

$$\begin{aligned}e(\beta', T) &= i_{\ell-1} - 1 < k \\ e(\rho_2, T) &= k - i_\ell < k \\ e(\sigma_1, T) &= i_\ell - 1 < k.\end{aligned}$$

This shows (a).

(3) $i_{\ell-1} > i_\ell$ and $i_{\ell+1} < i_\ell$. This case follows from the case (2) by symmetry.
(4) $i_{\ell-1} > i_\ell$ and $i_{\ell+1} > i_\ell$. This case follows from the case (1) by symmetry.

□

Let (S, M) be an unpunctured surface which is not simply connected. Let $T = \{\tau_1, \dots, \tau_N\}$ be a triangulation of S , and let γ be an arc. Choose an orientation of γ and let a be its starting point and b its endpoint. Denote by $\pi : \tilde{S} \rightarrow S$ a universal cover and define $\tilde{T} = \tau^{-1}(T)$.

Fix a point $\tilde{a} \in \pi^{-1}(a)$ and let $\tilde{\gamma}$ be the unique lift of γ starting at \tilde{a} . Although \tilde{T} is an infinite set, the set $\mathcal{P}_{\tilde{T}}(\tilde{\gamma})$ is finite, and we define the finite set

$$\overline{T} = \bigcup_{\tilde{\alpha} \in \mathcal{P}_{\tilde{T}}(\tilde{\gamma})} \{\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_{\ell(\tilde{\alpha})}\}.$$

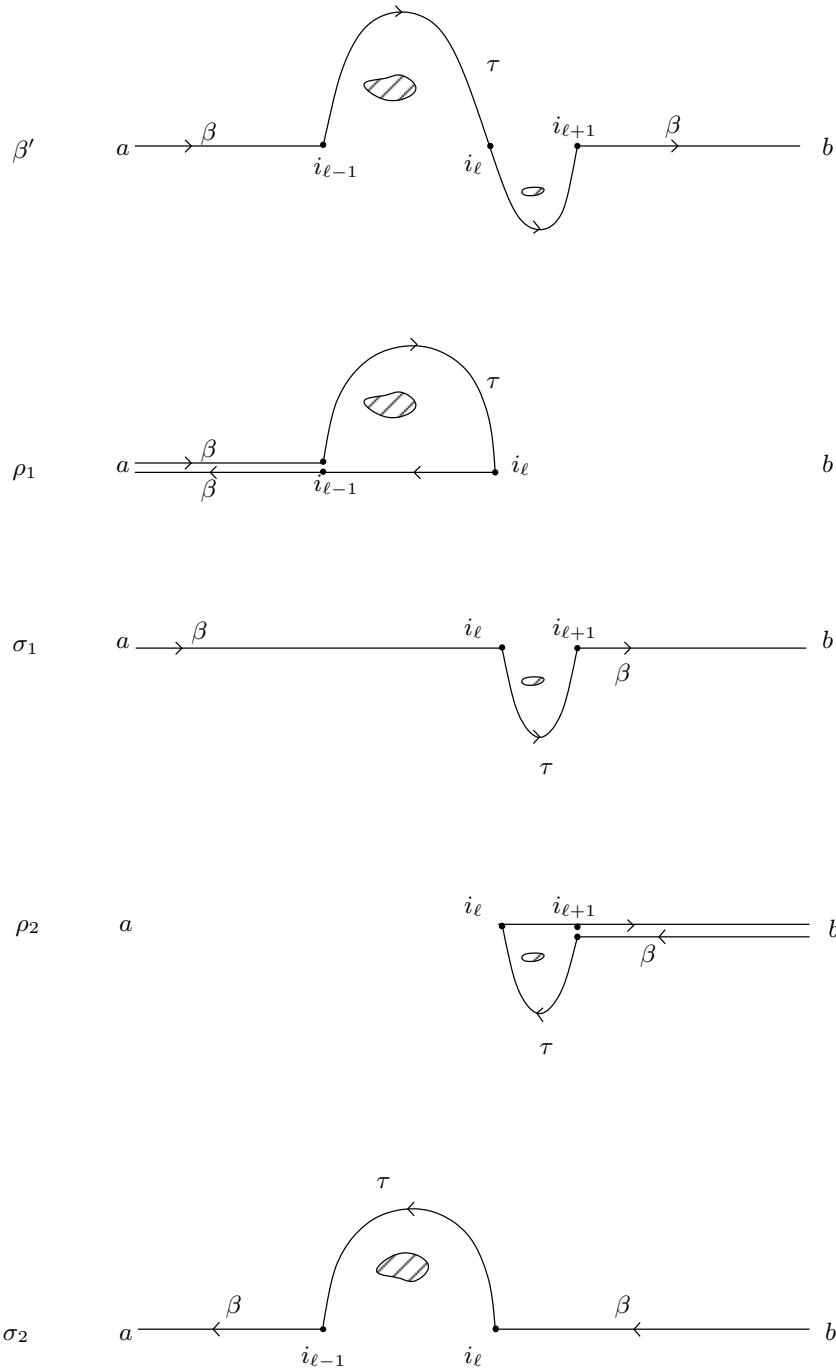


FIGURE 6. The arcs of Lemma 4.6 in case where $i_{\ell-1} < i_\ell$ and $i_{\ell+1} > i_\ell$.

Thus \overline{T} is the set of all arcs in \tilde{S} that occur in some path in $\mathcal{P}_{\tilde{T}}(\tilde{\gamma})$. Then \overline{T} is a triangulation of a simply connected surface $(\overline{S}, \overline{M})$: The boundary of \overline{S} is the union of all arcs that occur in some $\alpha \in \mathcal{P}_{\tilde{T}}(\tilde{\gamma})$ and do not cross γ , and the set of marked points $\overline{M} = \pi^{-1}(M) \cap \overline{S}$. Consider the cluster algebra $\mathcal{A}(\overline{S}, \overline{M})$. By Theorem 3.2 for simply connected surfaces, we have

$$(10) \quad x_{\tilde{\gamma}} = \sum_{\overline{\alpha} \in \mathcal{P}_{\overline{T}}(\tilde{\gamma})} x(\overline{\alpha}).$$

We want to apply π to this equation in order to finish the proof. Before we can do so, we need to study the effect of π on (T, γ) -paths and on cluster variables.

Lemma 4.7. *The covering map π induces a bijection*

$$\overline{\pi} : \mathcal{P}_{\overline{T}}(\tilde{\gamma}) \rightarrow \mathcal{P}_T(\gamma)$$

which sends a path $\overline{\alpha} = (\overline{\alpha}_1, \overline{\alpha}_2, \dots, \overline{\alpha}_{\ell(\overline{\alpha})})$ to the path $\overline{\pi}(\overline{\alpha}) = (\pi \circ \overline{\alpha}_1, \pi \circ \overline{\alpha}_2, \dots, \pi \circ \overline{\alpha}_{\ell(\overline{\alpha})})$.

Proof. $\overline{\pi}$ is well defined. Let $\overline{\alpha} \in \mathcal{P}_{\overline{T}}(\tilde{\gamma})$. First note that the number of crossings of $\tilde{\gamma}$ and \overline{T} equals the number of crossings of γ and T . Label the crossing points in \overline{S} by $\overline{1}, \overline{2}, \dots, \overline{k}$, such that $\pi(\overline{i}) = i$.

$\overline{\pi}(\overline{\alpha})$ is a T -path. Indeed $\pi \circ \overline{\alpha}_i \in T$, since $\overline{\alpha}_i \in \overline{T}$, and $s(\pi \circ \overline{\alpha}_i) = \pi(s(\overline{\alpha}_i)) = \pi(t(\overline{\alpha}_{i-1})) = t(\pi \circ \overline{\alpha}_{i-1})$, for $i = 2, 3, \dots, \ell(\alpha)$. Moreover, $s(\pi \circ \overline{\alpha}_1) = \pi(s(\overline{\alpha}_1)) = \pi(\tilde{a}) = a$, and $t(\pi \circ \overline{\alpha}_{\ell(\overline{\alpha})}) = \pi(t(\overline{\alpha}_{\ell(\overline{\alpha})})) = \pi(\tilde{b}) = b$.

$\overline{\pi}(\overline{\alpha})$ is reduced. Indeed, suppose $\tau = \pi \circ \overline{\alpha}_i = (\pi \circ \overline{\alpha}_{i-1})^-$. Then $(\tau, \tau^-) = \pi \circ (\overline{\alpha}_i, \overline{\alpha}_{i-1})$ is a loop in S which is homotopically trivial. Therefore, its lift $(\overline{\alpha}_i, \overline{\alpha}_{i-1})$ is a loop in \overline{S} , whence $\overline{\alpha}_i = (\overline{\alpha}_{i-1})^-$, a contradiction.

It remains to check the axioms (T1)–(T5) for $\overline{\pi}(\overline{\alpha})$.

(T1) $\ell(\overline{\pi}(\overline{\alpha})) = \ell(\overline{\alpha})$ is odd.

(T2) If i is even, $\overline{\alpha}_i$ crosses $\tilde{\gamma}$, hence $\pi \circ \overline{\alpha}_i$ crosses γ .

(T3) The number $e(\gamma, \tau_i)$ of crossings between γ and τ_i in S is equal to the number of crossings between $\tilde{\gamma}$ and $\pi^{-1}(\tau_i)$ in \overline{S} .

Now, $\pi \circ \overline{\alpha}_t = \tau_i$ if and only if $\overline{\alpha}_t \in \pi^{-1}(\tau_i) \cap \overline{S}$. Therefore, since $\overline{\alpha}$ satisfies condition (T3), the number of even integers t such that $\pi \circ \overline{\alpha}_t = \tau_i$ is at most the number of crossings between $\tilde{\gamma}$ and $\pi^{-1}(\tau_i)$ in \overline{S} , hence the number of even integers t such that $\pi \circ \overline{\alpha}_t = \tau_i$ is at most $e(\gamma, \tau_i)$.

(T4) Since $\overline{\alpha}$ satisfies condition (T4), there exists a subsequence $(\overline{i}_2, \overline{i}_4, \dots, \overline{i}_{\ell(\overline{\alpha})-1})$ of $(\overline{1}, \dots, \overline{k})$ of crossing points in \overline{S} such that the crossing point \overline{i}_j lies on $\overline{\alpha}_j$ and $\overline{i}_j \neq \overline{i}_\ell$ if $j \neq \ell$. Let i_j be the image of the crossing point \overline{i}_j under π . Then $(i_2, i_4, \dots, i_{\ell(\overline{\alpha})-1})$ is a subsequence of $(1, 2, \dots, k)$ in S such that the crossing point i_j lies on $\pi \circ \overline{\alpha}_{i_j}$ and $i_j \neq i_\ell$ if $j \neq \ell$.

(T5) Two paths α and β in S are homotopic if and only if their lifts $\tilde{\alpha}$ and $\tilde{\beta}$, which start at same point in \overline{S} , are homotopic. This implies (T5).

$\overline{\pi}$ is injective. Suppose $\overline{\pi}(\overline{\alpha}) = \overline{\pi}(\overline{\beta})$, that is, $\pi \circ \overline{\alpha}_i = \pi \circ \overline{\beta}_i$, for all i . In particular, $\ell(\overline{\alpha}) = \ell(\overline{\beta})$. Now, $\overline{\alpha}_i$ is the unique lift of $\pi \circ \overline{\alpha}_i$ that starts at $s(\overline{\alpha}_i)$, and $\overline{\beta}_i$ is the unique lift of $\pi \circ \overline{\beta}_i = \pi \circ \overline{\alpha}$ that starts at $s(\overline{\beta}_i)$, for $i = 1, \dots, \ell(\overline{\alpha})$. Moreover, $s(\overline{\alpha}_1) = \tilde{a} = s(\overline{\beta}_1)$. Consequently, $\overline{\alpha} = \overline{\beta}$ and $\overline{\pi}$ is injective.

$\overline{\pi}$ is surjective. For every $\alpha \in \mathcal{P}_T(\gamma)$ there is a lift $\tilde{\alpha}$ that starts at \tilde{a} . We have to show that $\tilde{\alpha} \in \mathcal{P}_{\overline{T}}(\tilde{\gamma})$. (T1) and (T2) follow directly from the construction, and (T5) holds since \overline{S} is simply connected.

Let $\alpha = (\alpha_1, \dots, \alpha_{\ell(\alpha)})$ and $\tilde{\alpha} = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_{\ell(\alpha)})$. Suppose there exist even integers s and t such that $\tilde{\alpha}_s = \tilde{\alpha}_t$. Then the arc $\alpha_s = \alpha_t$ crosses γ in the two points i_s and i_t . Suppose without loss of generality that $i_t < i_s$. Then the loop $\alpha^\circ = (i_s, i_t, i_s \mid \alpha_s, \gamma)$ is the image of a loop $\tilde{\alpha}^\circ = (\tilde{i}_s, \tilde{i}_t, \tilde{i}_s \mid \overline{\alpha}_s, \overline{\gamma})$ in \overline{S} under $\overline{\pi}$. Since $\pi : \tilde{S} \rightarrow S$ is a universal cover, α° is homotopically trivial, and therefore there is an arc in the isotopy class of α_s which crosses γ two times fewer than α_s , a contradiction.

This shows that the even arcs of $\tilde{\alpha}$ are pairwise disjoint. Therefore, $\tilde{\alpha}$ satisfies (T3) and (T4). \square

By the Laurent phenomenon [FZ1], every element of the cluster algebra $\mathcal{A}(\overline{S}, \overline{M})$ is a Laurent polynomial in the cluster variables $\{x_{\overline{\tau}} \mid \overline{\tau} \in \overline{T}\}$. Therefore, π induces a homomorphism of algebras $\pi_{\mathcal{A}} : \mathcal{A}(\overline{S}, \overline{M}) \rightarrow \mathcal{A}(S, M)$ which is given on the generators $x_{\overline{\tau}}$, $\overline{\tau} \in \overline{T}$ by

$$\pi_{\mathcal{A}}(x_{\overline{\tau}}) = x_{\pi \circ \overline{\tau}}.$$

Lemma 4.8. *Let $\tilde{\beta}$ be an arc in $(\overline{S}, \overline{M})$ which is a lift of an arc β in (S, M) . Then*

$$\pi_{\mathcal{A}}(x_{\tilde{\beta}}) = x_{\beta}$$

Proof. We prove the Lemma by induction on $k = e(\beta, T)$, the minimal number of crossing points between β and T . If $k = 0$, then $\beta \in T$, and $\pi_{\mathcal{A}}(x_{\tilde{\beta}}) = x_{\beta}$, by definition. Suppose that $k \geq 1$. Let ρ_1 , ρ_2 , σ_1 , σ_2 , and β' be as in Lemma 4.6. Then, in $\mathcal{A}(S, M)$, we have the exchange relation

$$(11) \quad x_{\beta} = (x_{\rho_1} x_{\rho_2} + x_{\sigma_1} x_{\sigma_2}) / x_{\beta'}.$$

Let $\tilde{\rho}_1$ and $\tilde{\sigma}_2$ be the unique lifts of ρ_1 and σ_2 , respectively, that start at the same point as $\tilde{\beta}$. Let $\tilde{\sigma}_1$ and $\tilde{\beta}'$ be the unique lifts of σ_1 and β' , respectively, that start at the endpoint of $\tilde{\rho}_1$, and let $\tilde{\rho}_2$ be the unique lift of ρ_2 that starts at the endpoint of $\tilde{\sigma}_2$. Then $\tilde{\rho}_1$, $\tilde{\rho}_2$, $\tilde{\sigma}_1$, $\tilde{\sigma}_2$ form a quadrilateral in \overline{S} in which $\tilde{\beta}$ and $\tilde{\beta}'$ are the diagonals. Consequently, in the cluster algebra $\mathcal{A}(\overline{S}, \overline{M})$, we have the exchange relation

$$x_{\tilde{\beta}} = (x_{\tilde{\rho}_1} x_{\tilde{\rho}_2} + x_{\tilde{\sigma}_1} x_{\tilde{\sigma}_2}) / x_{\tilde{\beta}'}.$$

Therefore

$$\pi_{\mathcal{A}}(x_{\tilde{\beta}}) = (\pi_{\mathcal{A}}(x_{\tilde{\rho}_1}) \pi_{\mathcal{A}}(x_{\tilde{\rho}_2}) + \pi_{\mathcal{A}}(x_{\tilde{\sigma}_1}) \pi_{\mathcal{A}}(x_{\tilde{\sigma}_2})) / \pi_{\mathcal{A}}(x_{\tilde{\beta}'}),$$

and by induction,

$$\pi_{\mathcal{A}}(x_{\tilde{\beta}}) = (x_{\rho_1} x_{\rho_2} + x_{\sigma_1} x_{\sigma_2}) / x_{\beta'},$$

which is equal to x_{β} by equation (11). \square

Proof of Theorem 3.2: By equation (10), we have

$$x_{\tilde{\gamma}} = \sum_{\overline{\alpha} \in \mathcal{P}_{\overline{T}}(\tilde{\gamma})} x(\overline{\alpha}).$$

We apply the homomorphism $\pi_{\mathcal{A}}$ to this equation, using Lemma 4.8, and we get

$$x_{\gamma} = \sum_{\overline{\alpha} \in \mathcal{P}_{\overline{T}}(\tilde{\gamma})} x(\overline{\pi}(\overline{\alpha})).$$

And by Lemma 4.7, this implies

$$x_\gamma = \sum_{\alpha \in \mathcal{P}_T(\gamma)} x(\alpha).$$

□

5. EXPANSION FORMULA USING MUTANT ARCS

For each non-boundary arc α of T , define α' to be the arc obtained by mutating T at α . We will call these “mutant arcs”. Let M be the set of these arcs.

Fix an arc γ which is not in T . In this section, we will give a new expression for x_γ .

Let X_γ be the set of arcs which γ crosses, or which bound a triangle which γ crosses. Let M_γ be the mutant arcs corresponding to arcs which γ crosses.

For every triangle through which γ passes, other than the first and last triangles, γ passes through two of the three arcs of the triangle. Let E_γ be the set of the third arcs of these triangles, that is to say, for each of these triangles, the arc through which γ does not pass.

We have already shown that x_γ can be written as a Laurent polynomial in x_σ for $\sigma \in X_\gamma$.

In this section, we will give an explicit expression for x_γ as a Laurent polynomial in x_σ for $\sigma \in X_\gamma \cup M_\gamma$, where only variables x_σ with $\sigma \in E_\gamma$ occur in the denominator.

5.1. Connection to Cluster Algebras. The motivation for this result is that it is an explicit version of a result from [BFZ1].

The matrix B arising from a cluster is called *acyclic* if it satisfies a certain condition, which, in our setting, is equivalent to the condition that every triangle of T has at least one arc lying along the boundary.

Define x'_i to be the cluster variable obtained by mutating x_i . Theorem 1.20 of [BFZ1] says that, if B is acyclic, any cluster variable in the cluster algebra has an expression as a polynomial $f \in \mathbb{ZP}[x_1, \dots, x_n, x'_1, \dots, x'_n]$.

Let us interpret this statement in our setting. Let E be the set of boundary arcs. Theorem 1.20 of [BFZ1] tells us precisely that if every triangle of T has an arc on the boundary, then there is an expression for an arbitrary x_γ as a Laurent polynomial in the variables x_σ with $\sigma \in T \cup M$, with only variables x_σ with $\sigma \in E$ appearing in the denominator. Since $E \supset E_\gamma$, $T \supset X_\gamma$, and $M \supset M_\gamma$, our result is an explicit realization of the kind of representation guaranteed by Theorem 1.20.

5.2. Recurrence. Let A, B be marked points, and let γ be an arc from A to B . As established in the previous section, if we want to calculate x_γ , we can “unwind” γ and reduce ourselves to the simply connected case, where A and B are marked points on the boundary of a polygon, P_{AB} , through all of whose triangles the line segment AB passes. We will now write E_{AB} instead of E_γ .

For any two marked points C and D on the boundary of P_{AB} , we write P_{CD} for the polygon consisting of the union of the triangles through which the line from C to D passes.

We will let I_{AB} be the set of interior arcs of P_{AB} . We will speak of the I_{AB} -degree of a vertex, meaning the number of I_{AB} -arcs incident with it.

Suppose that P_{AB} has at least five vertices. Let V_0 be the vertex adjacent to A whose T -degree is greater than one. Let V_1 be the other vertex adjacent to A , and let V_2 be the next vertex along the boundary from it, see Figure 7 or 8 for the two possible configurations.

Lemma 5.1. *We can calculate the cluster variable x_{AB} by means of the following recurrence:*

$$(12) \quad x_{AB} = \frac{x_{AV_2}x_{V_1B} - x_{AV_1}x_{V_2B}}{x_{V_1V_2}}$$

Proof. The result follows immediately from the exchange relation:

$$x_{AV_2}x_{V_1B} = x_{AV_1}x_{V_2B} + x_{AB}x_{V_1V_2}.$$

□

The above recursion is, in fact, all we need to recover Theorem 1.20 of [BFZ1] in our setting, as the following corollary shows.

Corollary 5.2. *There is an expression for x_{AB} as a Laurent polynomial in the variables x_σ for $\sigma \in X_{AB} \cup M_{AB}$, such that the only variables appearing in the denominator are x_σ with $\sigma \in E_{AB}$.*

Proof. The proof is by induction on the number of arcs of T crossed by AB . If AB crosses no arcs of T , it is in T , so x_{AB} is the desired expression. If AB crosses exactly one arc of T , then it is a mutant arc, so x_{AB} is, again, an expression of the desired form.

So assume that AB crosses at least two arcs of T , and that the claim is proved for any x_{CD} such that CD crosses fewer arcs of T than AB does. Since AB crosses at least two arcs of T , we know P_{AB} has at least five vertices, and we can apply Lemma 5.1. Let V_0, V_1, V_2 be as in the preamble to Lemma 5.1. So (12) holds. Note that the only variable in the denominator of the right hand side of (12) is $x_{V_1V_2}$, and V_1V_2 is in E_{AB} . Of the cluster variables which appear in the numerator, x_{AV_1} is from T , and x_{AV_2} is from M . This leaves x_{V_1B} and x_{V_2B} . The corresponding arcs are typically from neither X nor M , but they cross fewer arcs of T than does AB , so the induction hypothesis applies, and we know that x_{V_iB} can be written as a Laurent polynomial in x_σ with $\sigma \in T \cup M$, where the variables appearing in the denominator are from E_{V_iB} . Since $E_{V_iB} \subset E_{AB}$, the expression guaranteed by the induction hypothesis for x_{V_iB} is of the desired form, and the corollary is proved. □

5.3. Explicit formula. We will now proceed to give an explicit formula of the form guaranteed by Theorem 1.20 of [BFZ1] or by the previous corollary. In order to do this, we need to introduce some further notation. Number the arcs of I_{AB} as $I_{AB}^0, I_{AB}^1, \dots$, in the order in which they cross AB , and then likewise number the corresponding mutant arcs so that the mutation of the arc of I_{AB}^i is M_{AB}^i . The arc M_{AB}^i is considered to be oriented so that it crosses I_{AB}^i from the side on which A lies towards the side on which B lies. The boundary arcs of P_{AB} are oriented to point from A to B .

Let the arcs \bar{I}_{AB} be the arcs of I_{AB} which connect vertices whose I_{AB} -degree is at least two.

We now define a set of paths from A to B , which we denote \mathcal{G}_{AB} . These are the paths satisfying the following properties:

- G1_{AB} The length of the path is odd.
- G2_{AB} The arcs appearing are from the boundary of P_{AB} , \bar{T}_{AB} , and M_{AB} .
- G3_{AB} The arcs appearing in even position are all from E_{AB} .
- G4_{AB} The arcs of $\bar{T}_{AB} \cup M_{AB}$ are used in numerical order (so in particular, at most one of I_{AB}^i and M_{AB}^i is used).
- G5_{AB} The arcs of M_{AB} are used only in the forward direction.
- G6_{AB} The path should not touch A other than at the beginning, and not touch B other than at the end.

For γ an odd-length path between two vertices C and D , define $k(\gamma)$ to be the number of boundary arcs of γ which are used contrary to their orientation, plus $(\ell(\gamma) - 1)/2$, plus the number of arcs of I_{CD} used.

For \mathcal{A} a set of paths, write $\int \mathcal{A}$ for $\sum_{\gamma \in \mathcal{A}} (-1)^{k(\gamma)} x(\gamma)$. Now we have the following theorem:

Theorem 5.3. *We have the following expression for the cluster variable x_{AB} :*

$$(13) \quad x_{AB} = \int \mathcal{G}_{AB}$$

Proof. The proof is by induction. The theorem is clearly true if AB is an arc of T , or AB is a mutant arc. So assume that AB crosses at least two arcs of T , and that the theorem holds for any segment CD which crosses fewer arcs of T than AB does.

The proof involves showing that the formula (13) satisfies the recurrence in Lemma 5.1. By the induction hypothesis,

$$x_{V_i B} = \int \mathcal{G}_{V_i B}.$$

By Lemma 5.1, we therefore know:

$$x_{AB} = \frac{x_{AV_2}}{x_{V_2 V_1}} \int \mathcal{G}_{V_1 B} - \frac{x_{AV_1}}{x_{V_1 V_2}} \int \mathcal{G}_{V_2 B}.$$

We will now establish that the right hand side of the above equation equals $\int \mathcal{G}_{AB}$. We split into two cases, depending on whether the I_{AB} -degree of V_0 is 2 or more.

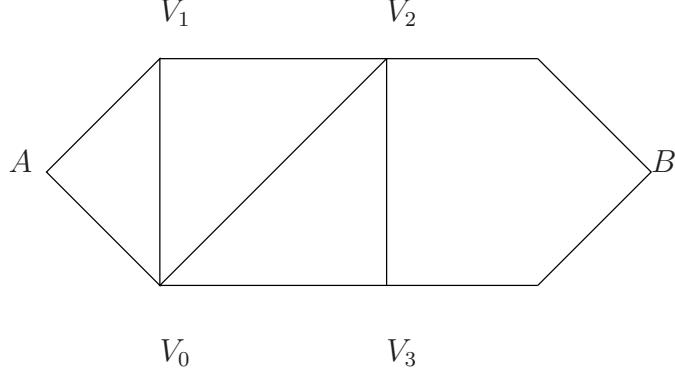
Suppose first that it is 2. So our situation is as in Figure 7. We have labelled the next vertex after V_0 as V_3 .

For $i = 1, 2, 3$, write \mathcal{F}_i for the paths from V_i to B but satisfying the conditions G1_{AB}–G6_{AB}. (In particular, no \mathcal{F}_i uses the arc $V_0 V_1$, since it is an internal arc not in \bar{T}_{AB} .) So $\mathcal{G}_{V_1 B}$ includes paths in addition to those in \mathcal{F}_1 , because $\mathcal{G}_{V_1 B}$ can include paths using the arc $V_0 V_1$.

Write $V_1 V_0 \mathcal{F}_3$ for paths that run from V_1 to V_0 to V_3 , and thence follow a path in \mathcal{F}_3 , and similarly for other (even-length) sequences of vertices followed by a set of paths. Then

$$\mathcal{G}_{V_1 B} = \mathcal{F}_1 \amalg V_1 V_0 \mathcal{F}_3.$$

Now,

FIGURE 7. First case, I_{AB} -degree of V_0 is 2

$$\frac{x_{AV_1}}{x_{V_1V_2}} \int \mathcal{G}_{V_2B} = - \int AV_1 \mathcal{G}_{V_2B}$$

where the negative sign appears because the lengths of the paths being summed over have each decreased by two. And similarly,

$$(14) \quad \frac{x_{AV_2}}{x_{V_2V_1}} \int \mathcal{G}_{V_1B} = \int AV_2 \mathcal{F}_1 - \int AV_2 V_1 V_0 \mathcal{F}_3$$

where the first term is positive because the change in length is cancelled out by adding a backwards boundary arc, and the negative sign in the second term appears because, in addition to the two effects already mentioned, we have an arc $V_1 V_0$ which was formerly a boundary arc, but is now in I_{AB} , so contributes a factor of -1.

We must simplify further, however, because the paths in the second term are not in \mathcal{G}_{AB} , as they use both $V_0 V_1$ and its mutant arc AV_2 , so this violates $G4_{AB}$. Thus, we must take advantage of the exchange relation which tells us that

$$x_{AV_2} x_{V_1 V_0} = x_{V_1 V_2} x_{AV_0} + x_{AV_1} x_{V_0 V_2}.$$

Thus,

$$(14) = \int AV_2 \mathcal{F}_1 + \int AV_1 V_2 V_0 \mathcal{F}_3 + \int AV_0 \mathcal{F}_3$$

Since

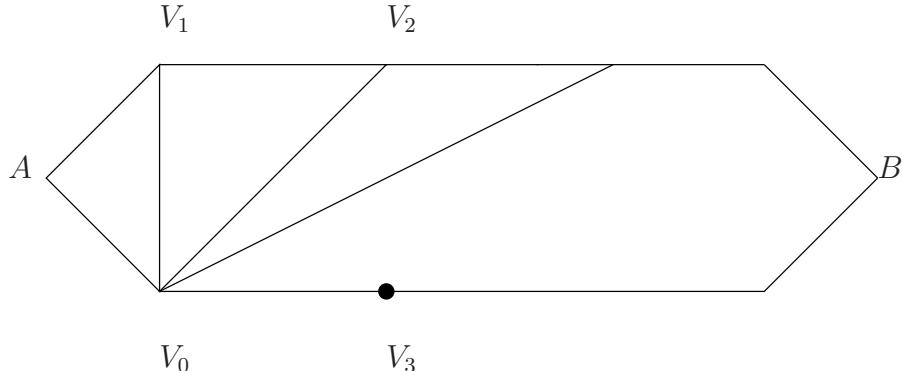
$$\mathcal{G}_{AB} = AV_2 \mathcal{F}_1 \amalg AV_1 V_2 V_0 \mathcal{F}_3 \amalg AV_0 \mathcal{F}_3 \amalg AV_1 \mathcal{G}_{V_2B},$$

it follows that $x_{AB} = \int \mathcal{G}_{AB}$, as desired.

We now consider the second case, where the situation is shown in Figure 8.

Note that the next vertex after V_0 away from A has been labelled V_3 . In this case, the analysis is similar to the previous case.

$$\begin{aligned} \mathcal{G}_{V_1B} &= \mathcal{F}_1 \amalg V_1 V_0 \mathcal{F}_3 \\ \mathcal{G}_{V_2B} &= \mathcal{F}_2 \amalg V_2 V_0 \mathcal{F}_3 \\ \frac{x_{AV_1}}{x_{V_1V_2}} \int \mathcal{G}_{V_2B} &= - \int AV_1 \mathcal{F}_2 + \int AV_1 V_2 V_0 \mathcal{F}_3 \end{aligned}$$

FIGURE 8. Second case, I_{AB} -degree of V_0 is greater than 2

$$\frac{x_{AV_2}}{x_{V_2V_1}} \int \mathcal{G}_{V_1B} = \int AV_2\mathcal{F}_1 - \int AV_2V_1V_0\mathcal{F}_3 = \int AV_2\mathcal{F}_1 + \int AV_1V_2V_0\mathcal{F}_3 + \int AV_0\mathcal{F}_3$$

Since $\mathcal{G}_{AB} = AV_2\mathcal{F}_1 \amalg AV_0\mathcal{F}_3 \amalg AV_1\mathcal{F}_2$, it follows that $x_{AB} = \int \mathcal{G}_{AB}$, as desired. \square

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